# MATH 111-007 LECTURE 13

Last lecture, we learned about the essence of the chain rule – a product of rates of changes when your parent function is a composite function, $f(g(x))$, namely,

$$\frac{d}{dx}f(g(x)) = f'(g(x))\,g'(x) = \frac{df}{dg}\cdot\frac{dg}{dx}.$$

It is imperative to identify $f$ and $g$ when you are given a function to differentiate.

We also learned that when dealing with functions composed with the exponential function, things become a little easier,

$$\frac{d}{dx}e^{g(x)} = e^{g(x)}g'(x).$$

**Example 1.** Differentiate $\sin\left(x^2+e^x\right)$.

**Solution.** Inner function is $g(x) = x^2+e^x$ and outer function is $f(x) = \sin(x)$ such that $\sin\left(x^2+e^x\right) = f(g(x))$. Thus,

$$\frac{d}{dx}\sin\left(x^2+e^x\right) = f'(g(x))\,g'(x) = \cos\left(x^2+e^x\right)(2x+e^x).$$

The Chain rule sometimes may be used repeatedly to find a derivative, when the parent function is complicated enough.

**Example 2.** Find the derivative of $g(t) = \tan(5-\sin(2t))$.

**Solution.** Inner function is $h(t) = 5-\sin(2t)$ and the outer function is $f(t) = \tan(t)$. We do see that $g(t) = f(h(t))$. Thus,

$$\begin{aligned} g'(t) &= f'(h(t))\,h'(t) \\ &= \sec^2(5-\sin(2t))\frac{d}{dt}(5-\sin(2t)) \\ &= \sec^2(5-\sin(2t))(-2\cos(2t)) \\ &= -2\cos(2t)\sec^2(5-\sin(2t)) \end{aligned}$$

Now, how about some pattern for powers of a function, that is, $h(x) = g(x)^n$? Note that the inner function is $g(x)$ while the outer function is in fact $f(x) = x^n$. Therefore,

$$\frac{dh}{dx} = h'(x) = f'(g(x))\,g'(x) = ng(x)^{n-1}g'(x).$$

**Example 3.** Use the above "Power Chain Rule" to find the derivative of $\left(5x^3-x^4\right)^7$.

**Solution.** Identify that $g(x) = 5x^3-x^4$ is the inner function, while the outer $f(x) = x^7$.

$$\begin{aligned} \frac{d}{dx}\left(5x^3-x^4\right)^7 &= 7\left(5x^3-x^4\right)^6\frac{d}{dx}\left(5x^3-x^4\right) \\ &= 7\left(5x^3-x^4\right)^6\left(15x^2-4x^3\right) \end{aligned}$$

I hate the quotient rule. But you cannot live without it until you learn both the Product Rule and the Chain Rule.

$$\begin{aligned} \frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) &= \frac{d}{dx}\left(u(x)\,v(x)^{-1}\right) \\ &= u'(x)\,v(x)^{-1} + u(x)\frac{d}{dx}\left(v(x)^{-1}\right) \\ &= u'(x)\,v(x)^{-1} + u(x)(-1)\,v(x)^{-2}v'(x) \\ &= \frac{u'(x)}{v(x)} - \frac{u(x)\,v'(x)}{v(x)^2} \end{aligned}$$

As a sanity check, you put the terms in the same denominator to retrieve the formula for the quotient rule,

$$\frac{u'(x)}{v(x)} - \frac{u(x)\,v'(x)}{v(x)^2} = \frac{u'(x)\,v(x)-u(x)\,v'(x)}{v(x)^2}.$$

Now, equipped with the Product rule, the Quotient rule, and the Chain rule, we may run into a function so complicated that all these rules are invoked when we try to compute its derivative. Let's go through some more complicated examples so you know what you would expect from an exam.

**Example 4.** Find the derivative of

(1) $y = \sin\left(x^2 e^x\right)$.

**Solution.** Inner function $g(x) = x^2 e^x$. Outer function $f(x) = \sin(x)$. Thus, we must do

$$\frac{dy}{dx} = f'(g(x))\, g'(x).$$

Don't hurry this. Let's tackle one term at a time since the functions involved are more complicated.

$$f'(x) = \cos(x) \implies f'(g(x)) = \cos\left(x^2 e^x\right).$$

Next,

$$g'(x) = \frac{d}{dx}\left(x^2 e^x\right) = 2xe^x + x^2 e^x = \left(x^2 + 2x\right) e^x.$$

Altogether,

$$\frac{dy}{dx} = \cos\left(x^2 e^x\right)\left(x^2 + 2x\right) e^x.$$

(2) $y = e^{x^2} + 5x$.

**Solution.** Only the first term poses a problem. For $e^{x^2}$, inner function is $g(x) = x^2$, outer function is $f(x) = e^x$. Thus,

$$\begin{aligned}\frac{dy}{dx} &= f'(g(x))\, g'(x) + 5 \\ &= e^{x^2}(2x) + 5.\end{aligned}$$

(3) $y = \left(t^{-3/4} \sin t\right)^{4/3}$.

**Solution.** Inner function is $g(t) = t^{-3/4} \sin t$ and outer function is $f(t) = t^{4/3}$. Therefore,

$$\frac{dy}{dt} = f'(g(t))\, g'(t)$$

where we tackle each term without a hurry.

$$f'(t) = \frac{4}{3} t^{\frac{1}{3}} \implies f'(g(t)) = \frac{4}{3}\left(t^{-3/4} \sin t\right)^{\frac{1}{3}} = \frac{4}{3} t^{-\frac{1}{4}} (\sin(t))^{\frac{1}{3}}.$$

Next,

$$g'(t) = -\frac{3}{4} t^{-\frac{7}{4}} \sin t + t^{-\frac{3}{4}} \cos t.$$

Altogether,

$$\begin{aligned}\frac{dy}{dt} &= \frac{4}{3} t^{-\frac{1}{4}} (\sin(t))^{\frac{1}{3}} \left(-\frac{3}{4} t^{-\frac{7}{4}} \sin t + t^{-\frac{3}{4}} \cos t\right) \\ &= -t^{-2} (\sin t)^{\frac{4}{3}} + \frac{4}{3} t^{-1} \cos t\, (\sin(t))^{\frac{1}{3}}.\end{aligned}$$

There are various other ways of posing a question involving the Chain rule. You won't always be asked to do the forward problem, which is given a function, find its derivative. You may get a question like: find $\frac{ds}{dt}$ when $\theta = \frac{3\pi}{2}$ if $s = \cos(\theta)$ and $\frac{d\theta}{dt} = 5$. This problem just involves that you use the Chain rule formula and solve for the unknown. First, by the chain rule,

$$\frac{ds}{dt} = \frac{ds}{d\theta} \cdot \frac{d\theta}{dt},$$

where each of these terms are functions – $\frac{ds}{d\theta}$ is a function of $\theta$, and $\frac{d\theta}{dt}$ is a function of $t$.

It seems that we already have the component $\frac{d\theta}{dt} = 5$. Great. We just need to find $\frac{ds}{d\theta}$ and evaluate it for $\theta = \frac{3\pi}{2}$.

$$\frac{ds}{d\theta} = \frac{d}{d\theta}(\cos\theta) = -\sin(\theta) \implies \frac{ds}{d\theta}\Big|_{\theta=\frac{3\pi}{2}} = -\sin\left(\frac{3\pi}{2}\right) = 1,$$

Altogether,

$$\frac{ds}{dt}\Big|_{\theta=\frac{3\pi}{2}} = \frac{ds}{d\theta}\Big|_{\theta=\frac{3\pi}{2}} \cdot 5 = 5.$$